Global small amplitude solutions for two-dimensional nonlinear Klein-Gordon systems in the presence of mass resonance

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Abstract: We consider a nonlinear system of two-dimensional Klein-Gordon equations with masses m_1, m_2 satisfying the resonance relation $m_2 = 2m_1 > 0$. We introduce a structural condition on the nonlinearities under which the solution exists globally in time and decays at the rate $O(|t|^{-1})$ as $t \to \pm\infty$ in L^∞. In particular, our new condition includes the Yukawa type interaction, which has been excluded from the *null condition* in the sense of J.-M.Delort, D.Fang and R.Xue (J.Funct.Anal.**211**(2004), 288–323).

Key Words: Nonlinear Klein-Gordon equations; Mass resonance; Global solution.

1   Introduction

This paper is intended to be a continuation of the papers [5], [6], [7], which are concerned with large time behavior of small solutions to the Cauchy problem for a nonlinear system of Klein-Gordon equations in $(t, x) \in \mathbb{R}^{1+2}$:

$$\begin{cases} (\Box + m_1^2)u_1 = F_1(u, \partial u), \\ (\Box + m_2^2)u_2 = F_2(u, \partial u), \end{cases} \tag{1.1}$$

where $\Box = \partial_t^2 - \partial_1^2 - \partial_2^2$, $\partial = (\partial_0, \partial_1, \partial_2)$ with $\partial_0 = \partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ for $j = 1, 2$, while $u = (u_j)_{j=1,2}$ is an $\mathbb{R}^2$-valued unknown function and $\partial u = (\partial_a u_j)_{\substack{j=1,2 \\ a=0,1,2}}$ is its first order derivative ($\mathbb{R}^{2\times3}$-valued). The masses m_1, m_2 are supposed to be positive constants. Without

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loss of generality, we may assume that $m_1 \leq m_2$ throughout this paper. The nonlinear term $F_j = F_j(v, w)$ is a C^∞ function of $(v, w) \in \mathbb{R}^2 \times \mathbb{R}^{2 \times 3}$ which vanishes at quadratic order at the origin, that is,

$$F_j(v, w) = O((|v| + |w|)^2) \quad \text{as} \quad (v, w) \to (0, 0).$$

For simplicity, the initial data are supposed to be of the form

$$u_j(0, x) = \varepsilon f_j(x), \quad \partial_t u_j(0, x) = \varepsilon g_j(x), \qquad x \in \mathbb{R}^2, \ j = 1, 2 \tag{1.2}$$

with a small parameter $\varepsilon > 0$ and C_0^∞ functions f_j, g_j.

From a perturbative point of view, quadratic nonlinear Klein-Gordon systems on $\mathbb{R}^2$ are of special interest because ratio of the masses and the structure of the nonlinearities play essential roles when one considers large time behavior of the solutions. Let us recall known results briefly. In the case of $m_2 \neq 2m_1$ (which will be referred to as the *non-resonant* case), it is shown in [5], [11] that the solution $u(t)$ for (1.1)–(1.2) exists globally without any structural restrictions of F_1, F_2 if ε is sufficiently small. Moreover, $u(t)$ is asymptotically free (in the sense that we can find a solution $u^\pm(t)$ of the homogeneous linear Klein-Gordon equations such that $u(t)$ tends to $u^\pm(t)$ as $t \to \pm\infty$ in the energy norm) and satisfies the following time decay estimate for all $p \in [2, \infty]$:

$$\sum_{|I| \leq 1} \|\partial_{t,x}^I u(t, \cdot)\|_{L^p(\mathbb{R}_x^2)} \leq C\varepsilon(1 + |t|)^{-\left(1 - \frac{2}{p}\right)} \qquad (t \in \mathbb{R}) \tag{1.3}$$

with some positive constant C which is independent of ε. Remember that this decay rate is same as that for the linear case. On the other hand, the above assertion fails to hold in the *resonant* case (i.e., the case where $m_2 = 2m_1$) because of counterexamples due to [6], [7], [10] etc. One of the simplest example is

$$\begin{cases} F_1 = 0 \\ F_2 = u_1^2. \end{cases}$$

For this nonlinearity, we can choose f_j, $g_j \in C_0^\infty(\mathbb{R}^2)$ and positive constants C, T such that the solution $u(t)$ for (1.1)–(1.2) satisfies

$$\sum_{|I| \leq 1} \|\partial_{t,x}^I u(t, \cdot)\|_{L^2} \geq C\varepsilon^2 \log |t| \qquad (|t| \geq T)$$

however small $\varepsilon > 0$ is, whence the estimate (1.3) is violated. Thus we need to put some structural condition on the nonlinearities in order to obtain global solutions for (1.1)–(1.2) satisfying (1.3) in the resonant case. This is what we are going to address here. A sufficient condition on the nonlinearities is introduced by Delort–Fang–Xue [2], called the *null condition*, which admits a global solution for (1.1)–(1.2) in the resonant case. They also give an

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asymptotic profile of the solution, from which the decay estimate (1.3) follows immediately. However, their condition is not optimal since it does not cover some important cases. For instance,

$$\begin{cases} F_1 = u_1 u_2 \\ F_2 = u_1^2 \end{cases} \tag{1.4}$$

is excluded from their condition, while the system (1.1) with the nonlinearity (1.4) can be viewed as a simplified model for some physical systems, such as Dirac-Klein-Gordon system, Maxwell-Higgs system, and so on. Someone may call this type of interaction the *Yukawa type* one (see e.g., [3], [11] and the references therein).

Our aim in this paper is to give a new sufficient condition on the nonlinearities which includes (1.4). Under this condition, we will show that the solution for (1.1)–(1.2) exists globally in time and it enjoys time decay property (1.3) even in the resonant case.

2   Main result

In order to state the result, let us introduce several notations. For $j = 1, 2$, denote by Q_j the quadratic homogeneous part of the nonlinear term F_j, that is,

$$Q_j(v, w) = \lim_{\lambda \downarrow 0} \lambda^{-2} F_j(\lambda v, \lambda w)$$

for $(v, w) \in \mathbb{R}^2 \times \mathbb{R}^{2 \times 3}$. Roughly saying, $Q_j(u, \partial u)$ gives the main part of the nonlinearity while $F_j(u, \partial u) - Q_j(u, \partial u)$ is regarded as a cubic or higher order remainder if we are interested in small amplitude solutions. Next we set

$$\mathbb{H} = \{\boldsymbol{\omega} = (\omega_0, \omega_1, \omega_2) \in \mathbb{R}^3 \ : \ \omega_0^2 - \omega_1^2 - \omega_2^2 = 1\}$$

and

$$\Phi_j(\boldsymbol{\omega}) = \int_0^1 Q_j\big(V(\theta), W(\boldsymbol{\omega}, \theta)\big) e^{-2\pi i j \theta} \, d\theta \tag{2.1}$$

for $\boldsymbol{\omega} \in \mathbb{H}$, where $V(\theta) = \big(\cos 2\pi k\theta\big)_{k=1,2}$, $W(\boldsymbol{\omega}, \theta) = \big(-\omega_a m_k \sin 2\pi k\theta\big)_{\substack{k=1,2 \\ a=0,1,2}}$ and $i = \sqrt{-1}$. Note that Φ_j can be explicitly computed only from m_1, m_2 and F_j. With these $\Phi_1(\boldsymbol{\omega})$ and $\Phi_2(\boldsymbol{\omega})$, we introduce the following two conditions:

(a) Both $\Phi_1(\boldsymbol{\omega})$ and $\Phi_2(\boldsymbol{\omega})$ vanish identically on $\mathbb{H}$.

(b) The real part of the product $\Phi_1(\boldsymbol{\omega})\Phi_2(\boldsymbol{\omega})$ is uniformly positive on $\mathbb{H}$, while the imaginary part of $\Phi_1(\boldsymbol{\omega})\Phi_2(\boldsymbol{\omega})$ vanishes identically on $\mathbb{H}$.

Our main result is the following theorem.

Theorem 2.1. *Let $m_2 = 2m_1 > 0$. Suppose that either the condition (a) or (b) is satisfied. Then (1.1)–(1.2) admits a unique global classical solution for sufficiently small ε. Moreover, for all $p \in [2, \infty]$, the solution $u(t)$ satisfies (1.3), i.e.,*

$$\sum_{|I| \leq 1} \|\partial_{t,x}^I u(t, \cdot)\|_{L^p(\mathbb{R}_x^2)} \leq C\varepsilon(1 + |t|)^{-\left(1 - \frac{2}{p}\right)}$$

with some positive constant C which does not depend on ε.

Remark 2.1. The condition (a) is equivalent to the null condition in the sense of [2]. On the other hand, the condition (b) is completely new, as far as the authors know. (1.4) is a typical example of the nonlinearity which is excluded from (a) but included in (b). As our proof below suggests, it may be reasonable to conjecture that the solution may *not* be asymptotically free under the condition (b) (while it is possible to prove that the solution is asymptotically free under the condition (a); see [4] for the detail). This problem will be discussed in a future work.

Remark 2.2. Our main result remains valid for quasilinear systems if the definition of Φ_j is slightly modified and a suitable hyperbolicity assumption is imposed on F_j.

The rest of this paper is organized as follows. In Section 3 we make some reduction of the problem along the idea of [1], [2] with a slight modification. Section 4 is devoted to the derivation of some energy inequalities. In Section 5 we specify the worst contribution of the nonlinearities in the resonant case. Section 6 describes a lemma on some ordinary differential equations, which reveals the role of our condition imposed on Φ_j. After that, we get an a priori estimate in Section 7, from which global existence follows immediately. Finally, in Section 8, the time decay estimate (1.3) is derived. In what follows, several positive constants appearing in estimates will be denoted by the same letter C, which may vary from line to line.

3   Reduction of the problem

In the following, we restrict ourselves to the forward Cauchy problem ($t > 0$) since the backward problem can be treated in the same way. Also, we shall neglect the higher order terms of F_j (i.e. we assume $F_j = Q_j$) to make the essential idea clearer.

Let K be a positive constant which satisfies

$$\operatorname{supp} f_j \cup \operatorname{supp} g_j \subset \big\{ x \in \mathbb{R} \ : \ |x| \leq K \big\}$$

and let τ_0 be a fixed positive number strictly greater than $1 + 2K$. We start with the fact that we may treat the problem as if the Cauchy data are given on the upper branch of the hyperbola

$$\left\{(t, x) \in \mathbb{R}^{1+2} \ : \ (t + 2K)^2 - |x|^2 = \tau_0^2, \ t > 0\right\}$$

and it is sufficiently smooth, small, compactly-supported. This is a consequence of the classical local existence theorem and the finite speed of propagation (see e.g., [1, Proposition 1.4] or [2, Proposition 1.1.4] for the detail). Next, let us introduce the hyperbolic coordinate $(\tau, z) \in [\tau_0, \infty) \times \mathbb{R}^2$ in the interior of the light cone, i.e.,

$$t + 2K = \tau \cosh |z|, \quad x_1 = \tau \frac{z_1}{|z|} \sinh |z|, \quad x_2 = \tau \frac{z_2}{|z|} \sinh |z|$$

for $|x| < t + 2K$. Then, with the auxiliary expression $z_1 = \rho \cos \theta$, $z_2 = \rho \sin \theta$, we see that

$$\begin{pmatrix} \partial_0 \\ \partial_1 \\ \partial_2 \end{pmatrix} = \begin{pmatrix} \partial_t \\ \partial_{x_1} \\ \partial_{x_2} \end{pmatrix} = \begin{pmatrix} \cosh \rho & -\sinh \rho & 0 \\ -\sinh \rho \cos \theta & \cosh \rho \cos \theta & -\sin \theta \\ -\sinh \rho \sin \theta & \cosh \rho \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \partial_\tau \\ \frac{1}{\tau} \partial_\rho \\ \frac{1}{\tau \sinh \rho} \partial_\theta \end{pmatrix},$$

whence

$$\partial_a = \omega_a(z) \partial_\tau + \frac{1}{\tau} \sum_{j=1}^{2} \eta_{aj}(z) \partial_{z_j} \tag{3.1}$$

for $a = 0, 1, 2$, where

$$\boldsymbol{\omega}(z) = \begin{pmatrix} \omega_0(z) \\ \omega_1(z) \\ \omega_2(z) \end{pmatrix} = \begin{pmatrix} \cosh \rho \\ -\sinh \rho \cos \theta \\ -\sinh \rho \sin \theta \end{pmatrix},$$

$$\begin{pmatrix} \eta_{01}(z) & \eta_{02}(z) \\ \eta_{11}(z) & \eta_{12}(z) \\ \eta_{21}(z) & \eta_{22}(z) \end{pmatrix} = \begin{pmatrix} -\sinh \rho \cos \theta & -\sinh \rho \sin \theta \\ \cosh \rho \cos^2 \theta + \frac{\rho}{\sinh \rho} \sin^2 \theta & \left(\cosh \rho - \frac{\rho}{\sinh \rho}\right) \cos \theta \sin \theta \\ \left(\cosh \rho - \frac{\rho}{\sinh \rho}\right) \cos \theta \sin \theta & \cosh \rho \sin^2 \theta + \frac{\rho}{\sinh \rho} \cos^2 \theta \end{pmatrix}.$$

Remark that $\omega_a(z)$ and $\eta_{bj}(z)$ can be regarded as C^∞ functions of $z \in \mathbb{R}^2$ which satisfy

$$|\omega_a(z)| + |\eta_{bj}(z)| \le C e^{|z|}$$

for $a, b = 0, 1, 2$ and $j = 1, 2$. Moreover, $\boldsymbol{\omega}(z) \in \mathbb{H}$ for all $z \in \mathbb{R}^2$. Also we observe that

$$\Box u = \frac{1}{\tau} \left(\partial_\tau^2 - \frac{1}{\tau^2} \Lambda_0\right) (\tau u),$$

where

$$\Lambda_0 = \partial_\rho^2 + \frac{\cosh \rho}{\sinh \rho}\partial_\rho + \frac{1}{\sinh^2 \rho}\partial_\theta^2. \tag{3.2}$$

Next we introduce a weight function $\chi(z) = e^{-\kappa\langle z\rangle}$ with a large parameter κ, where $\langle z\rangle = \sqrt{1 + |z|^2}$. (In fact, we shall not always need the explicit form of χ, but only the properties that χ is smooth, radial, as well as the estimates $0 < \chi(z) \le C_0 e^{-\kappa|z|}$ and $|\partial_z^I \chi(z)| \le C_I \chi(z)$ for any multi-indices I with some constants C_I. Another choice for such $\chi(z)$ may be $\frac{1}{\cosh(\kappa|z|)}$, as was done by Delort et al. in [1], [2]. We also note that $\kappa \ge 6$ is enough for our purpose.) With this weight function, let us define the new unknown function $v_j(\tau, z)$ by

$$u_j(t, x) = \frac{\chi(z)}{\tau}v_j(\tau, z).$$

Then we see that $v = (v_1, v_2)$ satisfies

$$\left(\partial_\tau^2 - \frac{1}{\tau^2}\Lambda + m_j^2\right)v_j = \tilde{Q}_j(\tau, z, v, \partial_{\tau,z}v)$$

if $u = (u_1, u_2)$ solves (1.1), where Λ is defined by

$$\Lambda v = e^{\kappa\langle z\rangle}\Lambda_0\left(e^{-\kappa\langle z\rangle}v\right)$$

and

$$\tilde{Q}_j(\tau, z, v, \partial_{\tau,z}v) = \frac{\chi(z)}{\tau}Q_j(v, \boldsymbol{\omega}(z)\partial_\tau v) + \sum_{\nu=0}^{1}\sum_{\substack{1\le k,l\le 2\\|I|\le 1,|J|\le\nu}}\frac{q_{\nu jklIJ}(z)}{\tau^{2+\nu}}\partial_z^I v_k \cdot \partial_\tau^{1-\nu}\partial_z^J v_l \tag{3.3}$$

with some $q_{\nu jklIJ} \in C^\infty(\mathbb{R}^2)$ satisfying

$$|\partial_z^L q_{\nu jklIJ}(z)| \le C_L e^{(2-\kappa)|z|}$$

for any multi-index L.

At last, the original problem (1.1)–(1.2) is reduced to

$$\begin{cases} \left(\partial_\tau^2 - \frac{1}{\tau^2}\Lambda + m_j^2\right)v_j = \tilde{Q}_j(\tau, z, v, \partial_{\tau,z}v), & \tau > \tau_0, \ z \in \mathbb{R}^2, \\ (v_j, \partial_\tau v_j)|_{\tau=\tau_0} = (\varepsilon\tilde{f}_j, \varepsilon\tilde{g}_j) & z \in \mathbb{R}^2, \end{cases} \tag{3.4}$$

where $\tilde{f}_j$ and $\tilde{g}_j$ are C^∞ functions of $z \in \mathbb{R}^2$ with compact support.

4 Commuting vector fields and energy inequalities

In this section, we will derive a kind of energy inequalities for the operator

$$P_m = \partial_\tau^2 - \frac{1}{\tau^2}\Lambda + m^2$$

with $m > 0$ which will be needed in Section 7. For this purpose it is helpful to introduce the following function class.

Definition 4.1. Let $\nu \in \mathbb{R}$. We denote by $\mathcal{S}^\nu$ the space of C^∞ functions $a(z)$ defined on $\mathbb{R}^2$ satisfying

$$\sup_{z \in \mathbb{R}^2} \left(\frac{|\partial_z^I a(z)|}{\langle z \rangle^{\nu - |I|}} \right) < \infty$$

for any multi-index I.

We start with splitting Λ into three parts: $\Lambda = \Lambda_0 + \Lambda_1 + \Lambda_2$, where Λ_0 is defined by (3.2) and

$$\Lambda_1 = -2\kappa \frac{\rho}{\langle \rho \rangle} \partial_\rho = -2\kappa \sum_{j=1}^{2} \frac{z_j}{\langle z \rangle} \partial_{z_j},$$

$$\Lambda_2 = \frac{\kappa^2 |z|^2 \langle z \rangle - \kappa}{\langle z \rangle^3} + \frac{\kappa |z| \cosh |z|}{\langle z \rangle \sinh |z|}.$$

Note that we can rewrite Λ_0 as

$$\Lambda_0 v = \frac{1}{\sqrt{\mathcal{G}(z)}} \sum_{i,j=1}^{2} \partial_{z_i} \left(\sqrt{\mathcal{G}(z)} g^{ij}(z) \partial_{z_j} v \right)$$

when we put

$$\begin{pmatrix} g^{11}(z) & g^{12}(z) \\ g^{21}(z) & g^{22}(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \left(\frac{1}{|z|^2} - \frac{1}{\sinh^2 |z|} \right) \begin{pmatrix} z_2^2 & -z_1 z_2 \\ -z_1 z_2 & z_1^2 \end{pmatrix}$$

and

$$\mathcal{G}(z) = \left(\frac{\sinh |z|}{|z|} \right)^2.$$

We observe that

$$\sum_{j,k=1}^{2} g^{jk}(z) \zeta_j \zeta_k = \left| \frac{z}{|z|} \cdot \zeta \right|^2 + \frac{1}{\sinh^2 |z|} |z \wedge \zeta|^2 \geq 0 \tag{4.1}$$

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for $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2$, and that

$$\mathcal{G}(z) = \det(g^{jk}(z))^{-1}_{1 \leq j,k \leq 2}.$$

Next, let us introduce the vector fields

$$\Gamma_1 = (t + 2K)\partial_{x_1} + x_1 \partial_t = (\cos\theta)\partial_\rho - \frac{\sin\theta}{\tanh\rho}\partial_\theta,$$

$$\Gamma_2 = (t + 2K)\partial_{x_2} + x_2 \partial_t = (\sin\theta)\partial_\rho + \frac{\cos\theta}{\tanh\rho}\partial_\theta,$$

$$\Gamma_3 = -x_2 \partial_{x_1} + x_1 \partial_{x_2} = \partial_\theta.$$

In what follows, we write $|I| = I_1 + I_2 + I_3$ and $\Gamma^I = \Gamma_1^{I_1} \Gamma_2^{I_2} \Gamma_3^{I_3}$ for a multi-index $I = (I_1, I_2, I_3)$. We can immediately check that

$$[\Gamma_1, \Gamma_2] = \Gamma_3, \quad [\Gamma_1, \Gamma_3] = \Gamma_2, \quad [\Gamma_2, \Gamma_3] = \Gamma_1,$$

where $[\cdot, \cdot]$ denotes the commutator. Another important thing is that Γ_1, Γ_2 are written as linear combinations of ∂_{z_1}, ∂_{z_2} with $\mathcal{S}^1$-coefficients, while ∂_{z_1}, ∂_{z_2} are written as linear combinations of Γ_1, Γ_2 with $\mathcal{S}^0$-coefficients. More precisely, we have

$$\Gamma_j = \sum_{k=1}^{2} c_{jk}(z)\partial_{z_k}$$

for $j = 1, 2$ and

$$\partial_{z_k} = \sum_{l=1}^{2} \tilde{c}_{kl}(z)\Gamma_l$$

for $k = 1, 2$, where

$$\begin{pmatrix} c_{11}(z) & c_{12}(z) \\ c_{21}(z) & c_{22}(z) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{\rho}{\tanh\rho} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

$$\begin{pmatrix} \tilde{c}_{11}(z) & \tilde{c}_{12}(z) \\ \tilde{c}_{21}(z) & \tilde{c}_{22}(z) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{\tanh\rho}{\rho} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

As for the commutation relation between P_m and Γ_j's, we have the following:

Lemma 4.1. *For any multi-index I, we have*

$$[P_m, \Gamma^I] = \frac{1}{\tau^2} \sum_{|J| \leq |I|} h_{IJ}(z)\Gamma^J$$

with some $h_{IJ} \in \mathcal{S}^0$.

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Proof: First we note that
$$[\Box + m^2, \Gamma^I] = 0$$
for any multi-index I, and that
$$\Box + m^2 = \partial_\tau^2 + \frac{2}{\tau}\partial_\tau - \frac{1}{\tau^2}\Lambda_0 + m^2 = P_m + \frac{1}{\tau^2}(\Lambda_1 + \Lambda_2).$$
So we have
$$[P_m, \Gamma^I] = -\frac{1}{\tau^2}[\Lambda_1 + \Lambda_2, \Gamma^I]$$
$$= \frac{1}{\tau^2}\sum_{j=1}^2 [p_j\Gamma_j, \Gamma^I] - \frac{1}{\tau^2}[\Lambda_2, \Gamma^I],$$
where $p_j(z) = 2\kappa\frac{z_j}{\langle z \rangle} \in \mathcal{S}^0$. By induction on I, we have the desired conclusion. $\qquad\square$

Now, we turn to the energy inequalities for the operator P_m which we need. For $s \in \mathbb{Z}_{\geq 0}$, we introduce the energy E_s as follows:
$$E_s(\tau; v, m) = \sum_{|I|\leq s}\frac{1}{2}\int_{\mathbb{R}^2}\left(\left(\partial_\tau\Gamma^I v\right)^2 + \frac{1}{\tau^2}\sum_{j,k=1}^2 g^{jk}(z)\left(\partial_{z_j}\Gamma^I v\right)\left(\partial_{z_k}\Gamma^I v\right) + m^2\left(\Gamma^I v\right)^2\right)\sqrt{\mathcal{G}(z)}dz.$$
We also introduce the norm $\|\cdot\|_{(s)}$ by
$$\|v\|_{(s)} := \sum_{|I|\leq s}\|\Gamma^I v\|_{L^2(\mathbb{R}^2;\sqrt{\mathcal{G}(z)}dz)}.$$

Lemma 4.2. *For $s \in \mathbb{Z}_{\geq 0}$, we have*
$$\frac{d}{d\tau}E_s(\tau; v, m) \leq \left(\frac{2\kappa}{\tau} + \frac{C}{\tau^2}\right)E_s(\tau; v, m) + CE_s(\tau; v, m)^{1/2}\|P_m v(\tau)\|_{(s)} \tag{4.2}$$
and
$$\frac{d}{d\tau}E_s(\tau; v, m) \leq \frac{C}{\tau^2}E_{s+1}(\tau; v, m) + CE_s(\tau; v, m)^{1/2}\|P_m v(\tau)\|_{(s)}. \tag{4.3}$$

Proof: First we consider the case of $s = 0$. As usual, we compute
$$\frac{d}{d\tau}E_0(\tau; v, m)$$
$$= \int_{\mathbb{R}^2}\left((\partial_\tau v)\partial_\tau^2 v + m^2 v\partial_\tau v + \frac{1}{\tau^2}\sum_{j,k=1}^2 g^{jk}(z)(\partial_{z_k}v)\partial_\tau\partial_{z_j}v - \frac{2}{\tau^3}\sum_{j,k=1}^2 g^{jk}(z)(\partial_{z_j}v)\partial_{z_k}v\right)\sqrt{\mathcal{G}(z)}dz$$
$$\leq \int_{\mathbb{R}^2}(\partial_\tau^2 v + m^2 v)(\partial_\tau v)\sqrt{\mathcal{G}(z)} - \frac{1}{\tau^2}\sum_{j,k=1}^2 \partial_{z_j}\left(\sqrt{\mathcal{G}(z)}g^{jk}(z)\partial_{z_k}v\right)(\partial_\tau v)dz$$
$$= \int_{\mathbb{R}^2}\left(P_m v - \frac{1}{\tau^2}\Lambda_1 v - \frac{1}{\tau^2}\Lambda_2 v\right)(\partial_\tau v)\sqrt{\mathcal{G}(z)}dz$$
$$\leq \|P_m v\|_{(0)}E_0(\tau; v, m)^{1/2} + \frac{1}{\tau^{1+l}}\int_{\mathbb{R}^2}\frac{|\Lambda_1 v|}{\tau^{1-l}}|\partial_\tau v|\sqrt{\mathcal{G}(z)}dz + \frac{C}{\tau^2}E_0(\tau; v, m)$$

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for $l = 0, 1$. We shall estimate the second term differently according to $l = 0$ or $l = 1$. In the case of $l = 0$, from the relations

$$|\Lambda_1 v| = \left| 2\kappa \frac{|z|}{\langle z \rangle} \partial_\rho v \right| \le 2\kappa |\partial_\rho v|$$

and

$$\sum_{j,k=1}^{2} g^{jk}(z)(\partial_{z_j} v)(\partial_{z_k} v) = |\partial_\rho v|^2 + \frac{1}{\sinh^2 |z|} |\partial_\theta v|^2 \ge |\partial_\rho v|^2$$

it follows that

$$
\begin{aligned}
\frac{1}{\tau} \int_{\mathbb{R}^2} \frac{|\Lambda_1 v|}{\tau} |\partial_\tau v| \sqrt{\mathcal{G}(z)} dz &\le \frac{2\kappa}{\tau} \int_{\mathbb{R}^2} \frac{|\partial_\rho v|}{\tau} |\partial_\tau v| \sqrt{\mathcal{G}(z)} dz \\
&\le \frac{\kappa}{\tau} \int_{\mathbb{R}^2} \left(\frac{|\partial_\rho v|^2}{\tau^2} + |\partial_\tau v|^2 \right) \sqrt{\mathcal{G}(z)} dz \\
&\le \frac{\kappa}{\tau} \int_{\mathbb{R}^2} \left(\frac{1}{\tau^2} \sum_{j,k=1}^{2} g^{jk}(z)(\partial_{z_j} v)(\partial_{z_k} v) + |\partial_\tau v|^2 \right) \sqrt{\mathcal{G}(z)} dz \\
&\le \frac{2\kappa}{\tau} E_0(\tau; v, m),
\end{aligned}
$$

which gives us (4.2) with $s = 0$. On the other hand, using the relation

$$|\Lambda_1 v||\partial_\tau v| = \left| 2\kappa \sum_{j=1}^{2} \frac{z_j}{\langle z \rangle} \Gamma_j v \right| |\partial_\tau v| \le \frac{\kappa}{m} \left(m^2 |\Gamma v|^2 + |\partial_\tau v|^2 \right),$$

we have

$$
\begin{aligned}
\frac{1}{\tau^2} \int_{\mathbb{R}^2} |\Lambda_1 v||\partial_\tau v| \sqrt{\mathcal{G}(z)} dz &\le \frac{C}{\tau^2} \int_{\mathbb{R}^2} \left(m^2 |\Gamma v|^2 + |\partial_\tau v|^2 \right) \sqrt{\mathcal{G}(z)} dz \\
&\le \frac{C}{\tau^2} E_1(\tau; v, m),
\end{aligned}
$$

which yields (4.3) with $s = 0$. Next we consider the case of $s \ge 1$. It follows from Lemma 4.1 that

$$\sum_{|I| \le s} \| [P_m, \Gamma^I] v \|_{(0)} \le \frac{C}{\tau^2} \| v \|_{(s)} \le \frac{C}{\tau^2} E_s(\tau; v, m)^{1/2}.$$

Therefore

$$\frac{d}{d\tau}E_s(\tau;v,m) = \sum_{|I|\leq s}\frac{d}{d\tau}E_0(\tau;\Gamma^I v,m)$$

$$\leq \sum_{|I|\leq s}\left\{\left(\frac{2\kappa}{\tau}+\frac{C}{\tau^2}\right)E_0(\tau;\Gamma^I v,m) + CE_0(\tau;\Gamma^I v,m)^{1/2}\|P_m\Gamma^I v(\tau)\|_{(0)}\right\}$$

$$\leq \left(\frac{2\kappa}{\tau}+\frac{C}{\tau^2}\right)E_s(\tau;v,m) + CE_s(\tau;v,m)^{1/2}\sum_{|I|\leq s}\left(\|\Gamma^I P_m v(\tau)\|_{(0)} + \|[P_m,\Gamma^I]v\|_{(0)}\right)$$

$$\leq \left(\frac{2\kappa}{\tau}+\frac{C}{\tau^2}\right)E_s(\tau;v,m) + CE_s(\tau;v,m)^{1/2}\|P_m v(\tau)\|_{(s)}.$$

This completes the proof of (4.2). In the same way (4.3) can be derived. $\square$

We close this section with the following lemma, which will be used in Section 7 to estimate quadratic terms.

Lemma 4.3. *For $\kappa > 9/2$ and $s \in \mathbb{Z}_{\geq 0}$, we have*

$$\|e^{-\kappa\langle z\rangle}\varphi\psi\|_{(s)} \leq C\left(\|e^{-2|z|}\varphi\|_{L^\infty}\|\psi\|_{(s)} + \|\varphi\|_{(s)}\|e^{-2|z|}\psi\|_{L^\infty}\right),$$

provided that the right hand side is finite.

Proof: First we note that

$$\sum_{|I|\leq s}|\Gamma^I\phi(z)|^2\sqrt{\mathcal{G}(z)} \leq C\sum_{|I|+j\leq s}\left|e^{(1/2)\langle z\rangle}\langle z\rangle^{|I|}\partial_z^I\partial_\theta^j\phi(z)\right|^2,$$

whence

$$\|\phi\|_{(s)} \leq C\sum_{|I|+j\leq s}\left\|e^{(1/2+\delta)\langle z\rangle}\partial_z^I\partial_\theta^j\phi\right\|_{L^2(\mathbb{R}^2;dz)}$$

for any $\delta > 0$. By taking $\delta = \kappa - 9/2$ (so that $1/2 + \delta = \kappa - 4$), we have

$$\|e^{-\kappa\langle z\rangle}\varphi\psi\|_{(s)}$$

$$\leq C\sum_{|I|+j\leq s}\left\|e^{(1/2+\delta)\langle z\rangle}\partial_z^I\partial_\theta^j(e^{-\kappa\langle z\rangle}\varphi\psi)\right\|_{L^2}$$

$$= C\sum_{|I|+j\leq s}\left\|e^{(\kappa-4)\langle z\rangle}\partial_z^I\left\{e^{-(\kappa-4)\langle z\rangle}\partial_\theta^j(e^{-2\langle z\rangle}\varphi\cdot e^{-2\langle z\rangle}\psi)\right\}\right\|_{L^2}$$

$$\leq C\sum_{|I|+j\leq s}\|\partial_z^I\partial_\theta^j(e^{-2\langle z\rangle}\varphi\cdot e^{-2\langle z\rangle}\psi)\|_{L^2}$$

$$\leq C\left\{\|e^{-2\langle z\rangle}\varphi\|_{L^\infty}\sum_{|I|+j\leq s}\|\partial_z^I\partial_\theta^j(e^{-2\langle z\rangle}\psi)\|_{L^2} + \|e^{-2\langle z\rangle}\psi\|_{L^\infty}\sum_{|I|+j\leq s}\|\partial_z^I\partial_\theta^j(e^{-2\langle z\rangle}\varphi)\|_{L^2}\right\}$$

$$\leq C\left\{\|e^{-2|z|}\varphi\|_{L^\infty}\sum_{|I|\leq s}\|\Gamma^I\psi\|_{L^2} + \|e^{-2|z|}\psi\|_{L^\infty}\sum_{|I|\leq s}\|\Gamma^I\varphi\|_{L^2}\right\}$$

$$\leq C\left\{\|e^{-2|z|}\varphi\|_{L^\infty}\|\psi\|_{(s)} + \|e^{-2|z|}\psi\|_{L^\infty}\|\varphi\|_{(s)}\right\}.$$

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$\square$

5 The leading part of the nonlinearity

The objective of this section is to extract the leading part of $Q_j(v, \boldsymbol{\omega}\partial_\tau v)$ under some assumptions on v. What we are going to prove is the following:

Lemma 5.1. *Let $m_2 = 2m_1 > 0$, $\boldsymbol{\omega} = (\omega_a)_{a=0,1,2} \in \mathbb{H}$, $T > \tau_0 > 0$ and $\varepsilon > 0$. Suppose that $v = (v_1, v_2)$ is an $\mathbb{R}^2$-valued function of $(\tau, z) \in [\tau_0, T) \times \mathbb{R}^2$ which satisfies*

$$|v_j(\tau, z)| + |\partial_\tau v_j(\tau, z)| \le C\varepsilon^{1/2}e^{2|z|}, \quad |(\partial_\tau^2 + m_j^2)v_j(\tau, z)| \le \frac{C\varepsilon^{1/2}e^{2|z|}}{\tau}$$

for $(\tau, z) \in [\tau_0, T) \times \mathbb{R}^2$, $j = 1, 2$. Then we have

$$\left| \frac{e^{-im_1\tau}}{\tau}Q_1(v, \boldsymbol{\omega}\partial_\tau v) - \left(\frac{\Phi_1(\boldsymbol{\omega})}{\tau}\overline{\alpha_1}\alpha_2 + \partial_\tau\gamma_1 \right) \right| \le \frac{C\varepsilon\langle\boldsymbol{\omega}\rangle^2 e^{4|z|}}{\tau^2}, \tag{5.1}$$

$$\left| \frac{e^{-im_2\tau}}{\tau}Q_2(v, \boldsymbol{\omega}\partial_\tau v) - \left(\frac{\Phi_2(\boldsymbol{\omega})}{\tau}\alpha_1^2 + \partial_\tau\gamma_2 \right) \right| \le \frac{C\varepsilon\langle\boldsymbol{\omega}\rangle^2 e^{4|z|}}{\tau^2}, \tag{5.2}$$

where $\Phi_j(\boldsymbol{\omega})$ is given by (2.1), α_j is defined by

$$\alpha_j(\tau, z) = e^{-im_j\tau}\left(1 + \frac{1}{im_j}\frac{\partial}{\partial\tau} \right)v_j(\tau, z), \tag{5.3}$$

$\overline{\alpha_j}$ denotes the complex conjugate of α_j, and γ_j is a function of $(\tau, z, \boldsymbol{\omega})$ satisfying

$$|\gamma_j(\tau, z, \boldsymbol{\omega})| \le \frac{C\varepsilon\langle\boldsymbol{\omega}\rangle^2 e^{4|z|}}{\tau},$$

for $(\tau, z, \boldsymbol{\omega}) \in [\tau_0, T) \times \mathbb{R}^2 \times \mathbb{H}$. In the above estimates, the constants C are independent of ε, T, τ, z, $\boldsymbol{\omega}$.

Proof: Because of the relations $v_k = \mathrm{Re}(\alpha_k e^{im_k\tau})$, $\omega_a\partial_\tau v_k = -\omega_a m_k\,\mathrm{Im}(\alpha_k e^{im_k\tau})$ and $m_2 = 2m_1$, we may regard $Q_j(v, \boldsymbol{\omega}\partial_\tau v)$ as a trigonometric polynomial in $e^{im_1\tau}$ (with coefficients depending on α_k, m_k, $\boldsymbol{\omega}$), that is,

$$Q_j(v, \boldsymbol{\omega}\partial_\tau v) = \sum_{\substack{1 \le k_1 \le k_2 \le 2 \\ \sigma_1, \sigma_2 \in \{+, -\}}} \Psi_{jk_1k_2}^{\sigma_1\sigma_2}(\boldsymbol{\omega})\alpha_{k_1}^{(\sigma_1)}\alpha_{k_2}^{(\sigma_2)}e^{i(\sigma_1 k_1 + \sigma_2 k_2)m_1\tau} \tag{5.4}$$

for $j = 1, 2$, where $\alpha_k^{(+)} = \alpha_k$, $\alpha_k^{(-)} = \overline{\alpha_k}$ and

$$\Psi_{jk_1k_2}^{\sigma_1\sigma_2}(\boldsymbol{\omega}) = \frac{m_1}{2\pi}\int_0^{2\pi/m_1} Q_j(\tilde{V}(\theta), \tilde{W}(\boldsymbol{\omega}, \theta))e^{-i(\sigma_1 k_1 + \sigma_2 k_2)m_1\theta}d\theta$$

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with $\tilde{V}(\theta) = \big(\cos km_1\theta\big)_{k=1,2}$, $\tilde{W}(\boldsymbol{\omega},\theta) = \big(-\omega_a m_k \sin km_1\theta\big)_{\substack{k=1,2 \\ a=0,1,2}}$. Now we focus on the relation $j = \sigma_1 k_1 + \sigma_2 k_2$, which implies creation of $e^{im_j\tau}$ in the right hand side of (5.4). We see that this relation is satisfied precisely when $(j,k_1,k_2,\sigma_1,\sigma_2) = (1,1,2,-,+)$ or $(2,1,1,+,+)$, and that

$$\Psi^{\sigma_1\sigma_2}_{jk_1k_2}(\boldsymbol{\omega}) = \begin{cases} \Phi_1(\boldsymbol{\omega}) & \text{if } (j,k_1,k_2,\sigma_1,\sigma_2) = (1,1,2,-,+), \\ \Phi_2(\boldsymbol{\omega}) & \text{if } (j,k_1,k_2,\sigma_1,\sigma_2) = (2,1,1,+,+). \end{cases}$$

This observation shows that

$$\frac{e^{-im_1\tau}}{\tau}Q_1(v,\boldsymbol{\omega}\partial_\tau v) - \frac{\Phi_1(\boldsymbol{\omega})}{\tau}\overline{\alpha_1}\alpha_2$$

and

$$\frac{e^{-im_2\tau}}{\tau}Q_2(v,\boldsymbol{\omega}\partial_\tau v) - \frac{\Phi_2(\boldsymbol{\omega})}{\tau}\alpha_1^2$$

are written as sums of the terms in the form

$$C(\boldsymbol{\omega})\alpha^{(\sigma_1)}_{k_1}\alpha^{(\sigma_2)}_{k_2}\frac{e^{i\mu\tau}}{\tau}$$

with some $\mu \in \mathbb{R}\backslash\{0\}$ and $C(\boldsymbol{\omega}) = O(\langle\boldsymbol{\omega}\rangle^2)$ $(|\boldsymbol{\omega}| \to \infty)$. Eventually we arrive at (5.1) and (5.2) through the identity

$$\alpha^{(\sigma_1)}_{k_1}\alpha^{(\sigma_2)}_{k_2}\frac{e^{i\mu\tau}}{\tau} = \frac{\partial}{\partial\tau}\left(\frac{\alpha^{(\sigma_1)}_{k_1}\alpha^{(\sigma_2)}_{k_2}e^{i\mu\tau}}{i\mu\tau}\right) - \frac{\partial}{\partial\tau}\left(\frac{\alpha^{(\sigma_1)}_{k_1}\alpha^{(\sigma_2)}_{k_2}}{\tau}\right)\frac{e^{i\mu\tau}}{i\mu}$$

combined with the estimates

$$|\alpha^{(\sigma)}_k| = |v_k| + \frac{1}{m_k}|\partial_\tau v_k| \le C\varepsilon^{1/2}e^{2|z|}$$

and

$$|\partial_\tau\alpha^{(\sigma)}_k| = \frac{1}{m_k}\big|\big(\partial_\tau^2 + m_k^2\big)v_k\big| \le \frac{C\varepsilon^{1/2}e^{2|z|}}{\tau}.$$

$\square$

6 A lemma on ODE

In this section we investigate the behavior as $\tau \gg \tau_0$ of the solution $(\beta_1(\tau,z),\beta_2(\tau,z))$ of

$$\begin{cases} i\dfrac{\partial\beta_1}{\partial\tau} = \dfrac{\chi_1(z)\Phi_1(\boldsymbol{\omega}(z))}{\tau}\overline{\beta_1}\beta_2 + r_1(\tau,z), \\ i\dfrac{\partial\beta_2}{\partial\tau} = \dfrac{\chi_2(z)\Phi_2(\boldsymbol{\omega}(z))}{\tau}\beta_1^{\,2} + r_2(\tau,z), \end{cases} \qquad \tau > \tau_0, \tag{6.1}$$

with the initial condition

$$\sup_{z\in\mathbb{R}^2}\Big(|\beta_1(\tau_0,z)| + |\beta_2(\tau_0,z)|\Big) \le C\varepsilon. \tag{6.2}$$

Here $\boldsymbol{\omega}(z) = (\cosh|z|, -z_1\frac{\sinh|z|}{|z|}, -z_2\frac{\sinh|z|}{|z|})$, Φ_j is given by (2.1), χ_j is a real-valued function satisfying

$$c \le \frac{\chi_1(z)}{\chi_2(z)} \le C$$

with some $C \ge c > 0$, and $r_j(\tau,z)$ satisfies

$$\sup_{z\in\mathbb{R}^2}|r_j(\tau,z)| \le \frac{C\varepsilon}{\tau^{2-\delta}}$$

with some $0 < \delta < 1$. Note that the condition (a) reduces the system (6.1) to a trivial one, that is to say $i\partial_\tau\beta_j = O(\varepsilon\tau^{-2+\delta})$, so it is easy to see that (β_1,β_2) stays bounded when τ becomes large. In the following, we will see that a bit weaker assertion is valid under the condition (b).

Lemma 6.1. *Suppose that the condition (b) is satisfied. Let (β_1,β_2) be the solution of (6.1)–(6.2) on $[\tau_0, T)$. Then we have*

$$\sup_{(\tau,z)\in[\tau_0,T)\times\mathbb{R}^2} e^{-2|z|}\Big(|\beta_1(\tau,z)| + |\beta_2(\tau,z)|\Big) \le C\varepsilon,$$

where C is independent of ε, T.

Proof: We first note that both $\Phi_1(\boldsymbol{\omega})$ and $\Phi_2(\boldsymbol{\omega})$ never vanish and that

$$|\Phi_1(\boldsymbol{\omega})\Phi_2(\boldsymbol{\omega})| = \mathrm{Re}\big(\Phi_1(\boldsymbol{\omega})\Phi_2(\boldsymbol{\omega})\big) = \Phi_1(\boldsymbol{\omega})\Phi_2(\boldsymbol{\omega}) \ge C_0$$

with some strictly positive constant C_0 by virtue of (b). We put

$$B_\varepsilon(\tau,z) = \Big(\lambda_1(z)|\beta_1(\tau,z)|^2 + \lambda_2(z)|\beta_2(\tau,z)|^2 + \varepsilon^2\Big)^{1/2}$$

with

$$\lambda_1(z) = e^{-2\langle z\rangle}\sqrt{\frac{\chi_2(z)|\Phi_2(\boldsymbol{\omega}(z))|}{\chi_1(z)|\Phi_1(\boldsymbol{\omega}(z))|}}, \quad \lambda_2(z) = e^{-2\langle z\rangle}\sqrt{\frac{\chi_1(z)|\Phi_1(\boldsymbol{\omega}(z))|}{\chi_2(z)|\Phi_2(\boldsymbol{\omega}(z))|}}.$$

Then we see that

$$\lambda_1(z) = e^{-2\langle z\rangle}\sqrt{\frac{\chi_2(z)}{\chi_1(z)}}\frac{|\Phi_2(\boldsymbol{\omega}(z))|}{\sqrt{\mathrm{Re}\big(\Phi_1(\boldsymbol{\omega})\Phi_2(\boldsymbol{\omega})\big)}} \le Ce^{-2|z|}(1 + |\boldsymbol{\omega}(z)|^2) \le C$$

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and
$$\lambda_2(z) = \frac{e^{-4\langle z \rangle}}{\lambda_1(z)} \geq Ce^{-4|z|}.$$

In the same way, we have $\lambda_2(z) \leq C$ and $\lambda_1(z) \geq Ce^{-4|z|}$. Therefore

$$B_\varepsilon(\tau, z) \geq Ce^{-2|z|}\Big(|\beta_1(\tau, z)| + |\beta_2(\tau, z)|\Big). \tag{6.3}$$

Next we observe that

$$\lambda_1(z)\chi_1(z)\Phi_1(\boldsymbol{\omega}(z)) = \lambda_2(z)\chi_2(z)\overline{\Phi_2(\boldsymbol{\omega}(z))},$$

which implies the matrix

$$\begin{pmatrix} \lambda_1(z) & 0 \\ 0 & \lambda_2(z) \end{pmatrix} \begin{pmatrix} 0 & \chi_1(z)\Phi_1(\boldsymbol{\omega}(z))\beta_1 \\ \chi_2(z)\Phi_2(\boldsymbol{\omega}(z))\overline{\beta_1} & 0 \end{pmatrix}$$

is hermitian. Thus, by rewriting (6.1) in the form

$$i\partial_\tau \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \frac{1}{\tau} \begin{pmatrix} 0 & \chi_1(z)\Phi_1(\boldsymbol{\omega}(z))\beta_1 \\ \chi_1(z)\Phi_2(\boldsymbol{\omega}(z))\overline{\beta_1} & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} r_1 \\ r_2 \end{pmatrix},$$

we see that

$$\begin{aligned}
B_\varepsilon(\tau, z)\partial_\tau B_\varepsilon(\tau, z) &= \frac{1}{2}\partial_\tau\Big(\lambda_1|\beta_1|^2 + \lambda_2|\beta_2|^2\Big) \\
&= \mathrm{Im}\left\{ \overline{(\beta_1 \ \ \beta_2)} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} i\partial_\tau \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \right\} \\
&= \mathrm{Im}\left\{ \overline{(\beta_1 \ \ \beta_2)} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \right\} \\
&\leq \frac{C\varepsilon}{\tau^{2-\delta}} B_\varepsilon(\tau, z).
\end{aligned}$$

Therefore we have

$$B_\varepsilon(\tau, z) \leq B_\varepsilon(\tau_0, z) + \int_{\tau_0}^{\infty} \frac{C\varepsilon}{\eta^{2-\delta}} d\eta \leq C\varepsilon,$$

which, together with (6.3), leads to the desired estimate. $\qquad\qquad\square$

7 A priori estimate

Now we are in a position to obtain an a priori estimate for the solution of (3.4), which is the main step of the proof of Theorem 2.1. We set

$$M(T) := \sup_{(\tau, z) \in [\tau_0, T) \times \mathbb{R}^2} e^{-2|z|}\Big(|v(\tau, z)| + |\partial_\tau v(\tau, z)| + \frac{1}{\tau}|\partial_z v(\tau, z)|\Big)$$

for the smooth solution $v = (v_1, v_2)$ to (3.4) on $\tau \in [\tau_0, T)$. We will prove the following:

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Lemma 7.1. *There exist $\varepsilon_1 > 0$ and $C_1 > 0$ such that $M(T) \leq \varepsilon^{1/2}$ implies $M(T) \leq C_1 \varepsilon$ for any $\varepsilon \in (0, \varepsilon_1]$. Here C_1 is independent of T.*

Once this lemma is proved, we can derive global existence of the solution in the following way: By taking $\varepsilon_0 \in (0, \varepsilon_1]$ so that $2C_1 \varepsilon_0^{1/2} \leq 1$, we deduce that $M(T) \leq \varepsilon^{1/2}$ implies $M(T) \leq \varepsilon^{1/2}/2$ for any $\varepsilon \in (0, \varepsilon_0]$. Then, by the continuity argument, we have $M(T) \leq C_1 \varepsilon$ as long as the solution exists. Therefore the local solution to (3.4) can be extended to the global one. Going back to the original variables, we deduce the small data global existence for (1.1)–(1.2).

The rest part of this section is devoted to the proof of Lemma 7.1. The proof will be divided into two steps: We first derive an auxiliary estimate for the energy

$$E_s(\tau) := E_s(\tau; v_1, m_1) + E_s(\tau; v_2, m_2)$$

under the assumption that $M(T) \leq \varepsilon^{1/2}$. Remark that we do not need the special structure of the nonlinearity at this stage. Next we will prove the improved estimate for $M(T)$ by using the condition (a) or (b).

7.1 Energy estimate with moderate growth

Our goal here is to show $E_{s_1}(\tau) \leq C\varepsilon^2 \tau^\delta$ under the assumption that $M(T) \leq \varepsilon^{1/2}$, where $s_1 \geq 4$ and $0 < \delta < 1$. We will argue along the same line as [8], [9]. We apply (4.2) with $s = s_0 + s_1 + 1$ at first, where s_0 is an integer greater than 2κ. Since Lemma 4.3 yields

$$\left\| \tilde{Q}_j(\tau, \cdot, v, \partial v) \right\|_{(s)} \leq \frac{C}{\tau} M(T) E_s(\tau)^{1/2} \leq \frac{C}{\tau} \varepsilon^{1/2} E_s(\tau)^{1/2},$$

we have

$$\frac{d}{d\tau} E_{s_0+s_1+1}(\tau) \leq \left(\frac{2\kappa + C\varepsilon^{1/2}}{\tau} + \frac{C}{\tau^2} \right) E_{s_0+s_1+1}(\tau) \leq \left(\frac{s_0 + \frac{1}{2}}{\tau} + \frac{C}{\tau^2} \right) E_{s_0+s_1+1}(\tau).$$

It follows from the Gronwall lemma that

$$E_{s_0+s_1+1}(\tau) \leq E_{s_0+s_1+1}(\tau_0) \exp\left(\int_{\tau_0}^{\tau} \frac{s_0 + \frac{1}{2}}{\eta} + \frac{C}{\eta^2} d\eta \right) \leq C\varepsilon^2 \tau^{s_0 + \frac{1}{2}}.$$

Next, we apply (4.3) with $s = s_0 + s_1$. Then we have

$$\frac{d}{d\tau} E_{s_0+s_1}(\tau) \leq \frac{C}{\tau^2} E_{s_0+s_1+1}(\tau) + \frac{C\varepsilon}{\tau} E_{s_0+s_1}(\tau) \leq C\varepsilon^2 \tau^{s_0 - \frac{3}{2}} + \frac{C\varepsilon}{\tau} E_{s_0+s_1}(\tau),$$

which yields

$$E_{s_0+s_1}(\tau) \leq C\varepsilon^2 \tau^{s_0 - \frac{1}{2}}.$$

Repeating this procedure recursively, we have

$$E_{s_0+s_1+1-n}(\tau) \le C\varepsilon^2\tau^{s_0-n+\frac{1}{2}}$$

for $n = 1, 2, \cdots, s_0$. Eventually we see that

$$E_{s_1+1}(\tau) \le C\varepsilon^2\tau^{1/2}.$$

Finally, we again use (4.3) with $s = s_1$ to obtain

$$\frac{d}{d\tau}E_{s_1}(\tau) \le \frac{C}{\tau^2}E_{s_1+1}(\tau) + \frac{C\varepsilon}{\tau}E_{s_1}(\tau) \le \frac{C\varepsilon^2}{\tau^{3/2}} + \frac{C\varepsilon}{\tau}E_{s_1}(\tau),$$

whence we deduce

$$E_{s_1}(\tau) \le C\varepsilon^2\tau^{C\varepsilon}$$

for $\tau \in [\tau_0, T]$. By choosing ε so small that $C\varepsilon \le \delta$, we arrive at the desired estimate.

7.2 Pointwise estimate

We are going to prove $M(T) \le C\varepsilon$. First we note that

$$|(\partial_\tau^2 + m_j^2)v_j| = \left|\tilde{Q}_j + \frac{1}{\tau^2}\Lambda v_j\right| \le \frac{Ce^{(6-\kappa)|z|}}{\tau}M(T)^2 + \frac{C}{\tau^2}E_4(\tau)^{1/2}.$$

This implies the assumption of Lemma 5.1 is satisfied if we take $\kappa \ge 6$ and $s_1 \ge 4$. Next we introduce $\alpha_j(\tau, z)$ by (5.3). Then we see that α_1 satisfies

$$i\partial_\tau\alpha_1 = \frac{e^{-im_1\tau}}{m_1}(\partial_\tau^2 + m_1^2)v_1 = \frac{\chi_1(z)\Phi_1(\boldsymbol{\omega}(z))}{\tau}\overline{\alpha_1}\alpha_2 + R_1 + \partial_\tau S_1,$$

where $\chi_1(z) = \chi(z)/m_1$, $S_1(\tau, z) = \chi_1(z)\gamma_1(\tau, z, \boldsymbol{\omega}(z))$ with γ_1 given by Lemma 5.1 and

$$R_1(\tau, z) = \frac{e^{-im_1\tau}}{m_1}\tilde{Q}_1 - \frac{\chi_1(z)\Phi_1(\boldsymbol{\omega}(z))}{\tau}\overline{\alpha_1}\alpha_2 - \partial_\tau S_1 + \frac{e^{-im_1\tau}}{m_1\tau^2}\Lambda v_1.$$

Since $\tilde{Q}_1$ is given by (3.3), it follows from (5.1) that

$$\begin{aligned}
|R_1(\tau, z)| \le & \frac{e^{-\kappa|z|}}{m_1}\left|\frac{e^{-im_1\tau}}{\tau}Q_1(v, \boldsymbol{\omega}(z)\partial_\tau v) - \left(\frac{\Phi_1(\boldsymbol{\omega}(z))}{\tau}\overline{\alpha_1}\alpha_2 + \partial_\tau\gamma_1\right)\right| \\
& + \frac{Ce^{(6-\kappa)|z|}}{\tau^2}M(T)^2 + \frac{C}{\tau^2}E_4(\tau)^{1/2} \\
\le & \frac{C\varepsilon\langle\boldsymbol{\omega}(z)\rangle^2 e^{(4-\kappa)|z|}}{\tau^2} + \frac{C\varepsilon e^{(6-\kappa)|z|}}{\tau^2} + \frac{C\varepsilon}{\tau^{2-\delta}} \\
\le & \frac{C\varepsilon}{\tau^{2-\delta}}
\end{aligned}$$

and

$$|S_1(\tau, z)| \leq \frac{C\varepsilon e^{(6-\kappa)|z|}}{\tau} \leq \frac{C\varepsilon}{\tau}.$$

Similarly we have

$$i\partial_\tau \alpha_2 = \frac{\chi_2(z)\Phi_2(\boldsymbol{\omega}(z))}{\tau}\alpha_1^2 + R_2 + \partial_\tau S_2$$

with $\chi_2(z) = \chi(z)/m_2$ and suitable R_2, S_2 satisfying

$$|R_2(\tau, z)| \leq \frac{C\varepsilon}{\tau^{2-\delta}}, \quad |S_2(\tau, z)| \leq \frac{C\varepsilon}{\tau}.$$

Now, we set $\beta_j = \alpha_j + iS_j$ so that (β_1, β_2) satisfies (6.1) with

$$r_1 = R_1 - \frac{\chi_1(z)\Phi_1(\boldsymbol{\omega}(z))}{\tau}\big(i\overline{\alpha}_1 S_2 - i\alpha_2 \overline{S}_1 + \overline{S}_1 S_2\big),$$

$$r_2 = R_2 - \frac{\chi_2(z)\Phi_2(\boldsymbol{\omega}(z))}{\tau}\big(2i\alpha_1 S_1 - S_1^2\big).$$

Since

$$|r_1| \leq |R_1| + C\frac{e^{(2-\kappa)|z|}}{\tau}\big(|\overline{\alpha}_1||S_2| + |\alpha_2||\overline{S}_1| + |\overline{S}_1||S_2|\big) \leq \frac{C\varepsilon}{\tau^{2-\delta}},$$

$$|r_2| \leq |R_2| + C\frac{e^{(2-\kappa)|z|}}{\tau}\big(|\alpha_1||S_1| + |S_1|^2\big) \leq \frac{C\varepsilon}{\tau^{2-\delta}},$$

we can apply Lemma 6.1 to obtain

$$\sup_{(\tau,z)\in[\tau_0,T)\times\mathbb{R}^2} e^{-2|z|}\big|\beta_j(\tau, z)\big| \leq C\varepsilon.$$

We thus deduce

$$\big|\alpha_j(\tau, z)\big| \leq \big|\beta_j(\tau, z)\big| + \big|S_j(\tau, z)\big| \leq C\varepsilon e^{2|z|}.$$

Finally, from

$$|v_j(\tau, z)| + |\partial_\tau v_j(\tau, z)| \leq C\big(|v_j(\tau, z)|^2 + m_j^2|\partial_\tau v_j(\tau, z)|^2\big)^{1/2} = C|\alpha_j(\tau, z)|$$

and

$$|\partial_z v_j(\tau, z)| \leq CE_3(\tau)^{1/2}$$

it follows that

$$M(T) \leq C\sup_{(\tau,z)\in[\tau_0,T)\times\mathbb{R}^2} e^{-2|z|}\Big(|\alpha(\tau, z)| + \frac{E_3(\tau)^{1/2}}{\tau}\Big) \leq C\varepsilon,$$

as desired. $\qquad\square$

8 End of the proof of Theorem 2.1

The remaining task is to show the decay estimate (1.3). Remember that our change of variables is

$$u_j(t, x) = \frac{\chi(z)}{\tau} v_j(\tau, z)$$

with

$$t + 2K = \tau \cosh |z|, \quad x_j = \tau \frac{z_j}{|z|} \sinh |z|$$

for $|x| < t + 2K$, and that $u(t, \cdot)$ is supported on $\{x \in \mathbb{R}^2 : |x| \leq t + K\}$. Moreover, we already know that $|v(\tau, z)| + |\partial_\tau v(\tau, z)| \leq C\varepsilon e^{2|z|}$ and $|\partial_z v(\tau, z)| \leq C\varepsilon \tau^{\delta/2}$. So it follows that

$$|u(t, x)| = \frac{\chi(z) \cosh |z|}{\tau \cosh |z|} |v(\tau, z)| \leq \frac{e^{-(\kappa-3)|z|}}{t + 2K} \cdot e^{-2|z|} |v(\tau, z)| \leq \frac{C\varepsilon}{1 + t}.$$

Also, by using (3.1), we see that

$$\partial_a u(t, x) = \frac{\omega_a(z)\chi(z)}{\tau} \partial_\tau v(\tau, z) + \frac{1}{\tau} \sum_{j=1}^{2} \chi(z)\eta_{aj}(z) \frac{\partial_{z_j} v(\tau, z)}{\tau}$$

$$+ \frac{1}{\tau^2} \left\{ -\chi(z)\omega_a(z) + \sum_{j=1}^{2} (\partial_{z_j}\chi(z))\eta_{aj}(z) \right\} v(\tau, z),$$

whence

$$\sum_{|I|=1} |\partial_{t,x}^I u(t, x)| \leq \frac{C\varepsilon e^{-(\kappa-3)|z|}}{t + 2K} + \frac{C\varepsilon e^{-(\kappa-5)|z|}}{(t + 2K)^2} \leq \frac{C\varepsilon}{1 + t}.$$

To sum up, we obtain (1.3) with $p = \infty$. As for the case of $p \in [2, \infty)$, we have

$$\sum_{|I| \leq 1} \|\partial_{t,x}^I u(t, \cdot)\|_{L^p(\mathbb{R}^2)} = \sum_{|I| \leq 1} \|\partial_{t,x}^I u(t, \cdot)\|_{L^p(\{x \in \mathbb{R}^2 ; |x| \leq t+K\})}$$

$$\leq C \sum_{|I| \leq 1} \|\partial_{t,x}^I u(t, \cdot)\|_{L^\infty} \cdot \left(\int_{\{x \in \mathbb{R}^2 : |x| < t+K\}} 1 \, dx \right)^{1/p}$$

$$\leq C\varepsilon (1 + t)^{-1+2/p},$$

which completes the proof. $\qquad\square$

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